A Do-It-Yourself Method for Finding $\sqrt{2}$

There are several algorithms for approximating the square root of a given positive number. One of the simplest algorithms, known as the "divide and average" method, was already known to the Babylonians. To approximate the value of $\sqrt{2}$, we start with an initial guess x_0 (which can be any positive number). From this initial guess we get a better approximation x_1 from the formula

$$x_1 = \frac{1}{2}(x_0 + 2/x_0)$$

We now repeat the process, obtaining a still better approximation x_2:

$$x_2 = \frac{1}{2}(x_1 + 2/x_1)$$

We continue in this manner until the desired accuracy is obtained. For example, starting with $x_0 = 1.5$, we obtain the approximations $x_1 = 1.4167$, $x_2 = 1.4142$, $x_3 = 1.4142$, . . . , so that already after two steps our result is correct to the nearest ten-thousandths.

It can be shown that the sequence

$$x_{i+1} = \frac{1}{2}(x_i + 2/x_i), \qquad i = 0,1,2, \ldots$$

converges to $\sqrt{2}$ regardless of the initial guess x_0. For example, starting with $x_0 = 10$—certainly a bad guess—we obtain the subsequent values $x_1 = 5.1$, $x_2 = 2.746$, $x_3 = 1.737$, $x_4 = 1.444$, $x_5 = 1.415$, $x_6 = 1.414$, ...

The method can easily be executed on any calculator; it is not even necessary to jot down each x_i in order to obtain the next approximation—we simply store it in the calculator's memory. The method can be extended to the square root of any positive number a: the sequence will be given by the formula

$$x_{i+1} = \frac{1}{2}(x_i + a/x_i), \qquad i = 0,1,2, \ldots$$

where the initial guess x_0 can be any positive number. The method is a special case of the Newton-Raphson approximation method. For further details, see any advanced calculus text.

Reference:

Maor, E., *To Infinity and Beyond*, Princeton University Press, NJ, 1991, page 49..

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